SDP bounds for distance-avoiding sets on compact spaces

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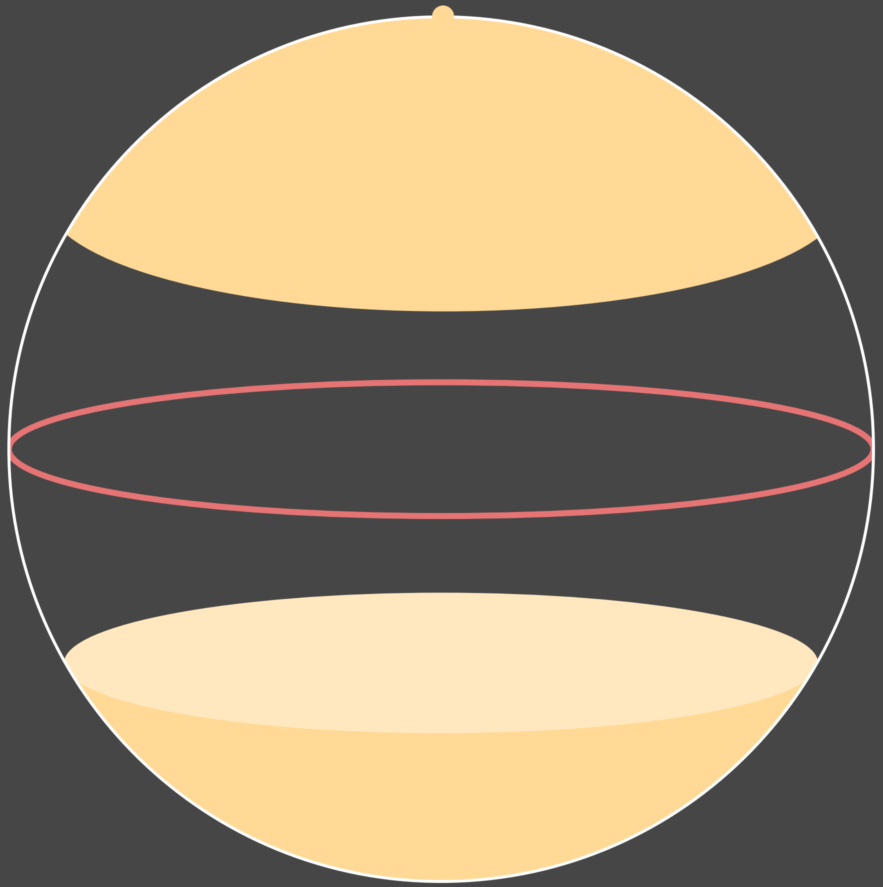

Witsenhausen's problem

What is the measure α_n of the largest measurable subset of S^{n-1} avoiding orthogonal points?

Double cap conjecture (Kalai; 2015)

The set $\{x \in S^{n-1} : |e \cdot x| > \cos(\pi/4)\}$ achieves α_n.

An upper bound is given by

ϑ-number

$$\vartheta = \sup \int_{S^{n-1}} \int_{S^{n-1}} A(x,y)\ dxdy$$

s.t.

- $\int_{S^{n-1}} A(x,x)\ dx = 1$
- $A(x,y) = 0$, if $x \cdot y = 0$
- $A \in C_{\text{sym}}(S^{n-1})_{\succcurlyeq 0}$

$A \succcurlyeq 0 \Leftrightarrow$ for all finite $U \subset S^{n-1}$

$$A[U] : U \times U \to \mathbb{R}$$

is positive semidefinite.

$S \subset S^{n-1}$ measurable avoiding orthogonal points.

Let

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S. \end{cases}$$

Define

$A'(x,y) = \chi_S \otimes \chi_S(x,y) = \chi_S(x)\chi_S(y).$

- A' is symmetric
- $A'(x,y) = 0$ if $x \cdot y = 0$
- $A' \succcurlyeq 0$
- $A(x,y) = \int_{O(n)} A'(Tx,Ty)\ dT$ is continuous
- $\int_{S^{n-1}} \int_{S^{n-1}} A(x,y)\ dxdy = \mathrm{Vol(S)}^2$
- $\int_{S^{n-1}} A(x,x)\ dx = \mathrm{Vol\,(S)}$

Lasserre's hierarchy

Consider continuous functions

$$A: \mathrm{sub}_k(S^{n-1}) \times \mathrm{sub}_k(S^{n-1}) \to \mathbb{R}$$

such that:

- $\int_{S^{n-1}} A(\{x\},\{x\})dx = 1$
- $A(S,T) = A(S',T') \quad \forall S \cup T = S' \cup T'$
- $A(S,T) = 0$ if $0 \in \{x \cdot y : x, y \in S \cup T\}$

This does not lead to a tractable upper bound for any $k \geq 2$

Completely and copositive cone

$$\mathrm{COP}(S^{n-1}) = \{A \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, f \otimes f \rangle \geq 0 \quad \forall f \in L^2(S^{n-1}), f \geq 0\}$$

$$\mathrm{CP}(S^{n-1}) = \mathrm{COP}(S^{n-1})^*$$

$$= \{A \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, Z \rangle \geq 0 \quad \forall Z \in \mathrm{COP}(S^{n-1})\}$$

Theorem (DeCorte, Oliveira, Vallentin)

$$\vartheta_{\mathrm{CP}(S^{n-1})} = \alpha_n$$

Optimizing over the completely positive cone is also not tractable

We approximate the copositive cone by cones

$$\mathcal{C}_r(S^{n-1}) = \{A \in L^2_{\text{sym}}(S^{n-1}) : \mathcal{R}_{S_{r+2}}(A \otimes 1^{\otimes r}) \geq 0\}$$

$A \in \mathcal{C}_r(S^{n-1})$, $0 \neq \mathrm{f} \geq 0$, then

$$\begin{aligned} 0 &\leq \langle \mathcal{R}_{S_{r+2}}(A \otimes 1^{\otimes r}), f^{\otimes(r+2)} \rangle \\ &= \langle A \otimes 1^{\otimes r}, f^{\otimes(r+2)} \rangle \\ &= \langle A \otimes 1^{\otimes r}, f \otimes f \otimes f^{\otimes r} \rangle \\ &= \langle A, f \otimes f \rangle \langle 1^{\otimes r}, f^{\otimes r} \rangle \end{aligned}$$

$$\mathcal{C}_1(S^{n-1}) \subseteq \mathcal{C}_2(S^{n-1}) \subseteq \cdots \subseteq \mathrm{COP}(\mathrm{S}^{\mathrm{n}-1})$$

Theorem (Bekker, Kuryatnikova, Oliveira, Vera)

If $A \in L^2_{\text{sym}}(S^{n-1})$ is such that $\langle A, Z \rangle \geq 0$ for all $Z \in \bigcup_{r \geq 1} \mathcal{C}_r(S^{n-1})$, then $A \in \mathrm{CP}(\mathrm{S}^{\mathrm{n}-1})$.

The copositive hierarchy

$$\vartheta_r = \sup \int_{S^{n-1}} \int_{S^{n-1}} A(x,y)\, dxdy$$

s.t.

- $\int_{S^{n-1}} A(x,x) dx = 1$
- $A(x,y) = 0$, if $x \cdot y = 0$
- $A \in \mathcal{C}_r(S^{n-1})^*,\ A \succcurlyeq 0$

Theorem (Bekker, Kuryatnikova, Oliveira, Vera)

If $n \geq 3,\ \lim_r(\vartheta_r) = \alpha_n$.

For any $n,\ r \geq 1$: $\text{lass}_{r+2} \leq \vartheta_r$.

Dimension	Lower bound	Previous best upper bound	New best upper bound	Percentage gap closed
3	0.2928	0.3015	0.2977	43%
4	0.1816	0.2168	0.1943	64%
5	0.1161	0.1677	0.1346	64%
6	0.0755	0.1338	0.0981	61%
7	0.0498	0.1174	0.0758	62%
8	0.0331	0.0998	0.0612	58%